Journal of Applied Mechanics and Technical Physics, Vol. 49, No. 1, pp. 129–135, 2008

MULTIMODULUS ELASTICITY THEORY

I. Yu. Tsvelodub

UDC 539.3

A variant of the multimodulus elasticity theory for isotropic materials is proposed under the assumption that the shear modulus in Hooke's law is a constant and the volume modulus depends on the sign of the first invariant of the stress tensor. Plane problems (plane strain and generalized plane stressed state) and problems of plate bending are considered. Some examples are given.

Key words: *multimodulus elasticity theory, three-constant elastic potential, plane problem, bending of multimodulus plates.*

The first postulates of the multimodulus elasticity theory (MMET) were developed in [1–4], which started a series of numerous publications continuing up to now. The papers published on this topic before 1980s were reviewed in [5, 6]. For the class of isotropic materials, the main problem in the MMET reduces to generalizing the classical elastic potential containing two constants (shear and volume moduli) and consistent with Hooke's law to media with different degrees of resistance to tension and compression. Various approaches were proposed in [1–6]; the number of independent elastic constants in these approaches varied from three to five (the maximum possible number): Young's moduli E_+ and E_-, Poisson's ratios under tension and compression ν_+ and ν_-, and shear modulus under pure shear G_0 (see also [7]). In [1, 2, 4, 7], the elastic potential contains the third invariant of the stress tensor in addition to the first and second invariants commonly used in the classical variant. In contrast to the approach developed in [8], this leads to emergence of tensor-nonlinear relations between stresses and strains. There is a current trend to construct tensor-linear constitutive equations of the MMET, based on three-constant potentials, independent of the third invariant [9, 10]. A simple variant of this theory is proposed in the present work; in this variant, the shear modulus G_0 is a constant, and the volume modulus K depends on the sign of the first invariant of the stress tensor. Such an approach allows plane problems in stresses to be reduced to formulations described in [11]. Problems of bending of multimodulus plates are also considered. A system of equations is derived for deflections and the function of membrane forces. An example of bending of a clamped elliptical plate is considered.

1. Simple Three-Constant MMET. For the classical isotropic medium that obeys Hooke's law, the elastic potential Φ, or the specific energy, can be presented as

$$\Phi = \Phi_1(I_\sigma) + \Phi_2(\sigma_i), \qquad \Phi_1 = \frac{1-2\nu}{6E}\, I_\sigma^2, \qquad \Phi_2 = \frac{1+\nu}{3E}\, \sigma_i^2, \tag{1.1}$$

where $I_\sigma = \sigma_{kk}$ is the first invariant, $\sigma_i^2 = (3/2)\sigma_{kl}^0\sigma_{kl}^0$, $\sigma_{kl}^0 = \sigma_{kl} - (1/3)I_\sigma\delta_{kl}$ $(k, l = 1, 2, 3)$, σ_i is the stress intensity, σ_{kl}, σ_{kl}^0, and δ_{kl} are the components of the stress tensor, stress deviator, and unit tensor, E is Young's modulus, and ν is Poisson's ratio; summation is performed over repeated indices from 1 to 3. The term $\Phi_1 = I_\sigma^2/(18K)$ is the specific work of volume changing and the term $\Phi_2 = \sigma_i^2/(6G)$ is the specific work of form changing; $K = E/[3(1-2\nu)]$ is the volume modulus of elasticity and $G = E/[2(1+\nu)]$ is the shear modulus.

A presentation similar to (1.1) of the elastic potential for multimodulus isotropic media in the form of the sum of the specific works of volume and form changing was proposed in [7], where Φ_1 depends on I_σ^2 and $\operatorname{sign} I_\sigma$, while Φ_2 is a function of the stress deviator invariants. The relations proposed describe the transition to the plastic state, where the material is plastically incompressible and the hydrostatic pressure, i.e., the value of I_σ, does not

Lavrent'ev Institute of Hydrodynamics, Siberian Division, Russian Academy of Sciences, Novosibirsk 630090; itsvel@hydro.nsc.ru. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 49, No. 1, pp. 157–164, January–February, 2008. Original article submitted February 2, 2007.

0021-8944/08/4901-0129

depend on the emergence of the plastic flow. Moreover, owing to the second term Φ_2 containing the function of the angle of the type of the stressed state ξ, which can be expanded into a series in powers of $\sin 3\xi$ by virtue of isotropy, the number N of independent elastic constants may be varied from three to five (as was noted above, these are the numbers of constants used in various MMET variants). For $N = 5$, the expression for Φ has the form [7]

$$\Phi = BI_\sigma^2 + (D_0 + D_1 \sin 3\xi + D_2 \sin^2 3\xi)^2 \sigma_i^2, \tag{1.2}$$

where

$$D_0 = (6G_0)^{-1/2}, \qquad D_1 = \frac{1}{2}\Big(\frac{1+\nu_-}{3E_-}\Big)^{1/2} - \frac{1}{2}\Big(\frac{1+\nu_+}{3E_+}\Big)^{1/2},$$

$$D_2 = \frac{1}{2}\Big(\frac{1+\nu_-}{3E_-}\Big)^{1/2} + \frac{1}{2}\Big(\frac{1+\nu_+}{3E_+}\Big)^{1/2} - (6G_0)^{-1/2},$$

$$B = \frac{1-2\nu_+}{6E_+}\, H(I_\sigma) + \frac{1-2\nu_-}{6E_-}\, H(-I_\sigma), \qquad H(x) = \begin{cases} 1, & x > 0, \\ 0, & x < 0, \end{cases} \qquad \sin 3\xi = -(9/2)\sigma_{kn}^0 \sigma_{nl}^0 \sigma_{kl}^0 / \sigma_i^3,$$

E_+ and E_- are Young's moduli, ν_+ and ν_- are Poisson's ratios under tension and compression, and G_0 is the shear modulus under pure shear.

It follows from (1.2) that $\Phi = \Phi(I_\sigma, \sigma_i, \xi)$ is a continuously differentiable function of its arguments, but $\partial^2\Phi/\partial I_\sigma^2 = 2B$ has a discontinuity at $I_\sigma = 0$. For $D_1 \neq 0$ or $D_2 \neq 0$, i.e., for the potential Φ possessing terms depending on the third invariant of the stress deviator, the corresponding relations between stresses and strains are tensor-nonlinear. As the objective of the present work is to construct a simple tensor-linear MMET, we confine ourselves to three constants in Eq. (1.2), assuming that $D_1 = D_2 = 0$. Introducing for convenience $c = B$ and $a = D_0^2$, we obtain the following relation from Eq. (1.2):

$$\begin{gathered} \Phi = a\sigma_i^2 + cI_\sigma^2, \qquad a = \frac{1+\nu_+}{3E_+} = \frac{1+\nu_-}{3E_-} = \frac{1}{6G_0}, \\ c = c_+ H(I_\sigma) + c_- H(-I_\sigma), \qquad c_\pm = (1-2\nu_\pm)/(6E_\pm). \end{gathered} \tag{1.3}$$

From Eq. (1.3), we obtain

$$\varepsilon_{kl} = \frac{\partial \Phi}{\partial \sigma_{kl}} = 3a\sigma_{kl}^0 + 2cI_\sigma \delta_{kl} \qquad (k, l = 1, 2, 3). \tag{1.4}$$

It should be noted that relations (1.3) and (1.4) can be obtained from a four-constant model [3], where the law of elasticity also depends on the sign of the first invariant I_σ: for $I_\sigma > 0$, relations (1.1) with the constants ν_+ and E_+ hold; for $I_\sigma < 0$, relations (1.1) with the constants ν_- and E_- are valid. For $I_\sigma = 0$, the law of elasticity is indeterminate, and the potential Φ gas a discontinuity. To eliminate this discontinuity, we need to impose the equality of the shear moduli $G_+ = E_+/[2(1+\nu_+)]$ and $G_- = E_-/[2(1+\nu_-)]$ under tension and compression; as a result, the model has three constants.

It follows from Eqs. (1.3) and (1.4) that $\varepsilon_{11} = \sigma/E_+$ and $\varepsilon_{22} = \varepsilon_{33} = -\nu_+\sigma/E_+$ under uniaxial tension (when the only stress component differing from zero is $\sigma_{11} = \sigma > 0$), $\varepsilon_{11} = \sigma/E_-$ and $\varepsilon_{22} = \varepsilon_{33} = -\nu_-\sigma/E_-$ under uniaxial compression ($\sigma_{11} = \sigma < 0$), and $2\varepsilon_{12} = \tau/G_0$ under torsion ($\sigma_{12} = \tau$).

Thus, relations (1.3) and (1.4) describe a multimodulus isotropic medium and are a simple generalization of the classical Hooke's law, where the shear modulus G is a constant and the volume modulus K depends on the sign of the first invariant I_σ of the stress tensor, i.e., the values of K under all-sided tension and compression are different.

The model constants a, c_+, and c_- are determined from the following experiments: 1) under pure shear, where the stress $\sigma_{12} = \tau$ is prescribed and the strain $\varepsilon_{12} = 3a\tau$ is measured; 2) under uniaxial tension ($\sigma_{11} = \sigma > 0$ and $I_\varepsilon = 6c_+\sigma$) and compression ($\sigma_{11} = \sigma < 0$ and $I_\varepsilon = 6c_-\sigma$), where the stress $\sigma_{11} = \sigma$ is prescribed and the volume strain $I_\varepsilon = \varepsilon_{kk}$ is measured.

Note, in contrast to the model developed in [1, 2], where the constitutive equations are written in the main axes of the stress tensor and the coefficients in these equations depend on the signs of the three main stresses, formulas (1.4), which are valid in an arbitrary Cartesian coordinate system, contain only one coefficient c depending on the sign of the invariant quantity I_σ (because $a = \mathrm{const}$). Despite the linearity of the relations between the main stresses and strains, the relations between the stress tensor and the strain tensor in [1, 2] are nonlinear [7].

2. Stability and Uniqueness of the Solution of MMET Boundary-Value Problems. The condition of stability in the small is formulated as follows [7]. For all infinitesimal increments of stresses $\delta\sigma_{kl}$ and the corresponding increments of strains $\delta\varepsilon_{kl}$, the following inequality is valid:

$$\delta\sigma_{kl}\delta\varepsilon_{kl} > 0, \qquad \delta\sigma_{kl}\delta\sigma_{kl} \neq 0. \tag{2.1}$$

For continuously differentiable functions $\varepsilon_{kl} = \varepsilon_{kl}(\sigma_{mn})$, this inequality is equivalent to a similar inequality for finite increments [12]

$$\Delta\sigma_{kl}\Delta\varepsilon_{kl} > 0, \quad \Delta\sigma_{kl}\Delta\sigma_{kl} \neq 0, \quad \Delta\sigma_{kl} = \sigma_{kl}^{(1)} - \sigma_{kl}^{(2)}, \quad \Delta\varepsilon_{kl} = \varepsilon_{kl}^{(1)} - \varepsilon_{kl}^{(2)}, \tag{2.2}$$

i.e., to the condition of stability in the large, which ensures the uniqueness of the solution of the boundary-value problems.

As was demonstrated in [7, 12], for inequality (2.1) to be satisfied for a potential of the general form $\Phi = \Phi(I_\sigma, \sigma_i, \xi)$, a necessary and sufficient condition is the positive determinacy of the matrix $\|a_{ij}\|$ with the coefficients

$$\begin{gathered} a_{11} = \frac{\partial^2\Phi}{\partial\sigma_i^2}, \qquad a_{22} = \frac{\partial\Phi}{\partial\sigma_i}\,\sigma_i + \frac{\partial^2\Phi}{\partial\xi^2}, \qquad a_{33} = \frac{\partial^2\Phi}{\partial I_\sigma^2}, \\ a_{12} = \sigma_i\,\frac{\partial}{\partial\sigma_i}\Big(\frac{1}{\sigma_i}\,\frac{\partial\Phi}{\partial\xi}\Big), \qquad a_{23} = \frac{\partial^2\Phi}{\partial I_\sigma\,\partial\xi}, \qquad a_{13} = \frac{\partial^2\Phi}{\partial I_\sigma\,\partial\sigma_i}. \end{gathered} \tag{2.3}$$

By virtue of relations (2.3), for the three-constant potential [10]

$$\Phi = a\sigma_i^2 + 2b\sigma_i I_\sigma + cI_\sigma^2,$$

the conditions of stability have the form of the inequalities

$$a > 0, \qquad ac - b^2 > 0, \qquad a\sigma_i + bI_\sigma > 0,$$

the latter imposing a constraint on the stressed state. This situation can be avoided by using a potential similar to that proposed in [9]:

$$\Phi = aI_2^2 + 2bI_2I_\sigma + cI_\sigma^2, \qquad I_2^2 = \sigma_{kl}\sigma_{kl}.$$

The conditions of stability that impose constraints on the elastic constants a, b, and c only were obtained in [13]. These constrains, however, are much more rigorous that the constrains for potential (1.3), which reduce to its positive determinacy [$a > 0$ and $c(\operatorname{sign} I_\sigma) > 0$]. Indeed, we obtain the following equation from (1.3):

$$\Delta\varepsilon_{kl}\Delta\sigma_{kl} = 3a\Delta\sigma_{kl}^0\Delta\sigma_{kl}^0 + 2\Delta(cI_\sigma)\Delta I_\sigma, \qquad \Delta I_\sigma = I_\sigma^{(1)} - I_\sigma^{(2)}. \tag{2.4}$$

If both stressed states refer to the region $I_\sigma > 0$ or $I_\sigma < 0$, the second term in (2.4) is $2c(\Delta I_\sigma)^2$. Then, with allowance for relations (2.2), we have $a > 0$, $c_+ > 0$, and $c_- > 0$ [this also follows from relations (2.3), because Φ from Eq. (1.3) in the regions considered is a twice continuously differentiable function of σ_i and I_σ]. For instance, for $I_\sigma^{(1)} > 0$ and $I_\sigma^{(2)} < 0$, we have $c^{(1)} = c_+$, $c^{(2)} = c_-$, and $\Delta(cI_\sigma)\Delta I_\sigma = [c_+I_\sigma^{(1)} - c_-I_\sigma^{(2)}][I_\sigma^{(1)} - I_\sigma^{(2)}] > 0$, because both terms are positive; for $I_\sigma^{(1)} < 0$ and $I_\sigma^{(2)} > 0$, we have $\Delta(cI_\sigma) < 0$ and $\Delta I_\sigma < 0$.

Thus, the condition of stability in the large (2.2), which guarantees the uniqueness of the solution of the boundary-value problems, is equivalent for the proposed potential (1.3) to the condition of satisfaction of the inequalities $a > 0$, $c_+ > 0$, and $c_- > 0$.

3. Plane MMET Problems. The advantage of the elasticity theory proposed is manifested in solving plane problems in stresses. Let us consider plane strain and the generalized plane stressed state and demonstrate that they are reduced to formulations used in [11].

In the case of plane strain, we assume that $\varepsilon_{33} = 0$ and find the following relation from Eq. (1.4):

$$\sigma_{33} = (a - 2c)(\sigma_{11} + \sigma_{22})/[2(a + c)].$$

Then, the expressions for the components ε_{kl} $(k, l = 1, 2)$ acquire the form

$$\begin{gathered} \varepsilon_{11} = a_1\sigma_{11} + a_2\sigma_{22}, \qquad \varepsilon_{22} = a_1\sigma_{22} + a_2\sigma_{11}, \\ \varepsilon_{12} = 3a\sigma_{12}, \quad a_1 = 3a(a + 4c)/[2(a + c)], \quad a_2 = 3a(2c - a)/[2(a + c)]. \end{gathered} \tag{3.1}$$

Introducing the function of stresses $F = F(x_1, x_2)$, such that $\sigma_{11} = F_{,22}$, $\sigma_{22} = F_{,11}$, and $\sigma_{12} = -F_{,12}$ (the subscript k after the comma means the derivative with respect to the coordinate x_k; $k = 1, 2$), and substituting these equalities into Eqs. (3.1), we obtain $\Delta\Delta F = 0$ from the condition of strain compatibility $\varepsilon_{11,22} + \varepsilon_{22,11} = 2\varepsilon_{12,12}$. This equality is valid for $c = c_+$ and for $c = c_-$, i.e., $F = F(x_1, x_2)$ is a biharmonic function both for $I_\sigma > 0$ and for $I_\sigma < 0$.

In the case of the generalized plane stressed state, Eqs. (1.4) yield relations of the form (3.1) for strains, where $a_1 = 2(a + c)$ and $a_2 = 2c - a$, and the equality $\Delta\Delta F = 0$ for F.

Thus, as in the classical plane problems, the stress function F in both cases is biharmonic; therefore, the methods of solving plane boundary-value problems in stresses are similar to the methods described in [11]. This is fairly understandable, because the stressed state of, at least, a simply connected solid under fixed external loads is independent of the elastic constants [11], i.e., substitution of the constant c_- for the constant c_+ in Eqs. (1.3) does not affect the stress distribution.

According to (3.1), the strains can be determined from the stresses found. In both cases, we have $\operatorname{sign} I_\sigma = \operatorname{sign}(\sigma_{11}+\sigma_{22})$, because $I_\sigma = \sigma_{11}+\sigma_{22}+\sigma_{33} = 3a(\sigma_{11}+\sigma_{22})/[2(a+c)]$ in the case of plane strain, and $I_\sigma = \sigma_{11}+\sigma_{22}$ in the generalized plane stressed state.

4. Problems of Plate Bending. Let us consider a plate of constant thickness h, which is deformed under the action of bending moments and (or) surface loads distributed along the plate edges; the upper layers of the plate become extended, and the lower layers of the plate become compressed ($I_\sigma < 0$ for $-h/2 \le z < \delta h/2$ and $I_\sigma > 0$ for $\delta h/2 < z \le h/2$; $\delta = \delta(x_1, x_2)$, where $|\delta| < 1$). The coordinate system is chosen so that the mid-plane of the plate coincides with the plane Ox_1x_2, and the z axis is perpendicular to the latter.

The strains and displacements of the plate are related as [6]

$$\varepsilon_{kl} = z\varkappa_{kl} + \varepsilon^0_{kl}, \qquad 2\varepsilon^0_{kl} = u_{k,l} + u_{l,k}, \qquad \varkappa_{kl} = -w_{,kl}, \tag{4.1}$$

where $\varkappa_{kl}$ are the curvatures, u_k are the displacements of the mid-plate, and w is the deflection.

If there are no tangential components of the external load, the equilibrium equations have the form [6]

$$Q_k = M_{kl,l}, \qquad Q_{k,k} = -q, \qquad N_{kl,l} = 0, \qquad Q_k = \int\limits_{-h/2}^{h/2} \sigma_{3k}\, dz,$$

$$M_{kl} = \int\limits_{-h/2}^{h/2} \sigma_{kl} z\, dz, \qquad N_{kl} = \int\limits_{-h/2}^{h/2} \sigma_{kl}\, dz, \tag{4.2}$$

where Q_k and M_{kl} are the shear forces and moments, N_{kl} are the membrane forces, and q is the intensity of surface loads. In Eqs. (4.1) and (4.2) and further, we have $k, l = 1, 2$.

For $a_1 = 2(a + c)$ and $a_2 = 2c - a$, the relations between the stresses and strains have the form (3.1). Inverting these relations, we obtain

$$\begin{gathered}\sigma_{11} = A\varepsilon_{11} + B\varepsilon_{22}, \qquad \sigma_{22} = A\varepsilon_{22} + B\varepsilon_{11}, \qquad \sigma_{12} = \varepsilon_{12}/(3a),\\ A = 2(a + c)/[3a(a + 4c)], \qquad B = (a - 2c)/[3a(a + 4c)].\end{gathered} \tag{4.3}$$

Taking into account that the function of the form $f = f[z, A(\operatorname{sign} I_\sigma)]$ $[A = A_+H(I_\sigma) + A_-H(-I_\sigma)$, where $A_\pm = A(c_\pm)]$ obeys the equality

$$\int\limits_{-h/2}^{h/2} f\, dz = \int\limits_{-h/2}^{\delta h/2} f(z, A_-)\, dz + \int\limits_{\delta h/2}^{h/2} f(z, A_+)\, dz,$$

we find the following expressions for N_{kl} and M_{kl} from Eqs. (4.1)–(4.3):

$$N_{11} = A_1\varepsilon^0_{11} + B_1\varepsilon^0_{22} + C(\varkappa_{11} + \varkappa_{22}), \qquad N_{22} = A_1\varepsilon^0_{22} + B_1\varepsilon^0_{11} + C(\varkappa_{11} + \varkappa_{22}),$$

$$N_{12} = h\varepsilon^0_{12}/(3a), \qquad M_{11} = C(\varepsilon^0_{11} + \varepsilon^0_{22}) + A_2\varkappa_{11} + B_2\varkappa_{22},$$

$$M_{22} = C(\varepsilon_{11}^0 + \varepsilon_{22}^0) + A_2\varkappa_{22} + B_2\varkappa_{11}, \qquad M_{12} = h^3\varkappa_{12}/(36a),$$
$$2A_1 = h(P - 2\theta\delta), \qquad 2B_1 = (Q - 2\theta\delta), \tag{4.4}$$
$$4C = h^2\theta(1-\delta^2), \qquad 24A_2 = h^3(P - 2\theta\delta^3), \qquad 24B_2 = h^3(Q - 2\theta\delta^3),$$
$$P = A_+ + A_- = 2G_0[(1-\nu_+)^{-1} + (1-\nu_-)^{-1}], \qquad Q = B_+ + B_- = P - 4G_0,$$
$$2\theta = A_+ - A_- = B_+ - B_- = 2G_0[(1-\nu_+)^{-1} - (1-\nu_-)^{-1}].$$

Here the equalities from (1.3) were used for a, c_+, and c_-.

We divide the membrane forces, moments, and curvatures by $Ph/2$, $Ph^2/12$, and $2/h$, respectively, and obtain their dimensionless values with a zero superscript. Then, with allowance for the equality $\delta = -(\varepsilon_{11}^0 + \varepsilon_{22}^0)/(\varkappa_{11}^0 + \varkappa_{22}^0)$ following from Eqs. (4.1) and (4.3) and the condition $I_\sigma = 0$ for $z = \delta h/2$, formulas (4.4) acquire the form

$$N_{11}^0 = \varepsilon_{11}^0 + \alpha\varepsilon_{22}^0 + r(1+\delta^2)(\varkappa_{11}^0 + \varkappa_{22}^0),$$
$$N_{22}^0 = \varepsilon_{22}^0 + \alpha\varepsilon_{11}^0 + r(1+\delta^2)(\varkappa_{11}^0 + \varkappa_{22}^0), \qquad N_{12} = (1-\alpha)\varepsilon_{12}^0,$$
$$M_{11}^0 = \varkappa_{11}^0 + \alpha\varkappa_{22}^0 + r(3-\delta^2)(\varepsilon_{11}^0 + \varepsilon_{22}^0), \qquad M_{12}^0 = (1-\alpha)\varkappa_{12}^0,$$
$$M_{22}^0 = \varkappa_{22}^0 + \alpha\varkappa_{11}^0 + r(3-\delta^2)(\varepsilon_{11}^0 + \varepsilon_{22}^0), \qquad \delta = -(\varepsilon_{11}^0 + \varepsilon_{22}^0)/(\varkappa_{11}^0 + \varkappa_{22}^0), \tag{4.5}$$
$$\alpha = Q/P = 1 - 2(1-\nu_+)(1-\nu_-)/(2-\nu_+-\nu_-) \qquad (0 < \alpha < 1),$$
$$r = \theta/P = (\nu_+ - \nu_-)/[2(2-\nu_+-\nu_-)].$$

The constant r from Eqs. (4.5) can be considered as a small parameter. Then, the function $\delta = \delta(x_1, x_2)$ defining the displacement of the neutral surface ($I_\sigma = 0$) from the mid-plane ($z = 0$) is a quantity of the order of r. At least, this fact is valid for the problem of pure bending of a rectangular plate by moments M_1 and M_2 uniformly distributed along the plate edges. It follows from the equilibrium equations (4.2) that $N_{kl} = 0$, $M_{11} = M_1$, $M_{22} = M_2$, and $M_{12} = 0$ at all points (x_1, x_2). Then, we find $\varepsilon_{11}^0 = \varepsilon_{22}^0$ from Eqs. (4.5); for the displacement δ, we obtain the equation $\delta^2 - 2\beta\delta + 1 = 0$, where $\beta = (1+\alpha)/(4r)$. Thus, β and r are quantities of the same sign; hence, we have $|\delta| = (|\beta| + \sqrt{\beta^2 - 1})^{-1} \le |\beta|^{-1} = 4|r|/(1+\alpha)$, i.e., $\delta \sim r$. The solution for δ exists if $|\beta| \ge 1$. Based on the known value of δ, we can easily find the curvatures $\varkappa_{kl}^0$ and strains ε_{kl}^0 from Eqs. (4.5).

Neglecting terms that contain the factor $r\delta^2$ (i.e., a quantity of the order of r^3) in Eqs. (4.5), we find the linear relations between the forces (N_{kl}^0 and M_{kl}^0) and kinematic quantities (ε_{kl}^0 and $\varkappa_{kl}^0$). Then, the first three equations in (4.5) yield

$$(1-\alpha^2)\varepsilon_{11}^0 = N_{11}^0 - \alpha N_{22}^0 - (1-\alpha)r\varkappa^0,$$
$$(1-\alpha^2)\varepsilon_{22}^0 = N_{22}^0 - \alpha N_{11}^0 - (1-\alpha)r\varkappa^0, \qquad \varkappa^0 = \varkappa_{11}^0 + \varkappa_{22}^0, \tag{4.6}$$
$$(1-\alpha)\varepsilon_{12}^0 = N_{12}^0,$$

and the last three equations have the form

$$M_{11}^0 = \varkappa_{11}^0 + \alpha\varkappa_{22}^0 + 3r\varepsilon^0, \qquad M_{22}^0 = \varkappa_{22}^0 + \alpha\varkappa_{11}^0 + 3r\varepsilon^0,$$
$$M_{12} = (1-\alpha)\varkappa_{12}^0, \qquad \varepsilon^0 = \varepsilon_{11}^0 + \varepsilon_{22}^0. \tag{4.7}$$

Introducing the function of the membrane forces $F^0 = F^0(x_1, x_2)$, such that $N_{11}^0 = F_{,22}^0$, $N_{22}^0 = F_{,11}^0$, and $N_{12}^0 = -F_{,12}^0$, taking into account the equalities $\varkappa_{kl}^0 = -w_{,kl}^0$ ($w_{,kl}^0 = hw_{,kl}/2$ are dimensionless quantities), and substituting Eq. (4.6) into the equation of strain compatibility $\varepsilon_{11,22}^0 + \varepsilon_{22,11}^0 - 2\varepsilon_{12,12}^0 = 0$, we obtain

$$\Delta\Delta[F^0 + r(1-\alpha)w^0] = 0. \tag{4.8}$$

From Eqs. (4.7) and the equilibrium equation $M_{kl,kl}^0 = -q^0$ ($q^0 = 12q/(Ph^2)$), we find

$$\Delta\Delta w^0 - 3r\Delta\varepsilon^0 = q^0. \tag{4.9}$$

It follows from Eqs. (4.6) that

$$\varepsilon^0 = [N_{11}^0 + N_{22}^0 - 2r(\varkappa_{11}^0 + \varkappa_{22}^0)]/(1+\alpha) = \Delta(F^0 + 2rw^0)/(1+\alpha).$$

Substituting this expression into Eq. (4.9), we obtain

$$\Delta\Delta[(1+\alpha-6r^2)w^0 - 3rF^0] = (1+\alpha)q^0. \tag{4.10}$$

Thus, we obtain system (4.8), (4.10) to find the functions $F^0 = F^0(x_1, x_2)$ and $w^0 = w^0(x_1, x_2)$.

As an example, let us consider the problem of an elliptical plate clamped over the contour γ and subjected to a uniformly distributed surface load $q = \text{const}$. There are no external forces applied to γ in the plane of the plate, i.e., $N_{kl}n_l = 0$ $(k = 1, 2)$; n_k are the components of the unit vector of the external normal to γ. Hence, we obtain the following boundary conditions on the contour γ defined by the equation $x_1^2 b_1^{-2} + x_2^2 b_2^{-2} = 1$:

$$w^0 = \frac{\partial w^0}{\partial n} = F^0 = \frac{\partial F^0}{\partial n} = 0. \tag{4.11}$$

The solution of the boundary-value problem (4.8), (4.10), (4.11) is sought in the form

$$\begin{gathered} F^0 = A_3\varphi(x_1, x_2), \qquad w^0 = B_3\varphi(x_1, x_2), \\ \varphi(x_1, x_2) = (x_1^2 b_1^{-2} + x_2^2 b_2^{-2} - 1)^2; \end{gathered} \tag{4.12}$$

conditions (4.11) are satisfied automatically thereby. Substituting Eqs. (4.12) into (4.8) and (4.10), we obtain two equalities, which yield the constants A_3 and B_3:

$$A_3 = -r(1-\alpha)q^0/C_1, \quad B_3 = q^0/C_1, \quad C_1 = 3(1-3r^2)[8(b_1^{-4} + b_2^{-4}) + (b_1 b_2)^{-2}].$$

5. Some Generalizations to More Complicated Media. Preserving the tensor-nonlinear relations between the stresses and strains (or strain rates), we can generalize this approach to physically nonlinear media with different properties under tension and compression. Examples of such media are nonlinearly elastic and nonlinearly viscous media. The strain potentials in the first case or the strain-rate potential in the second case is a uniform function of stresses of power $n+1$ $(n > 1)$. As is implied above, Eq. (1.3) with the right side risen to the power $(n+1)/2$ can be used as such a function.

Let us consider a nonlinearly viscous medium. The dissipative function W, which is the specific power of dissipated energy and differs from the creep potential only by a constant factor $(n+1)$ [12], is defined as follows:

$$W = B_0(\sigma_i^2 + c^0 I_\sigma^2)^{(n+1)/2}, \quad W \equiv \eta_{kl}\sigma_{kl}, \quad c_0 = c_+^0 H(I_\sigma) + c_-^0 H(-I_\sigma) \tag{5.1}$$

[η_{kl} are the components of the strain rates; the function $H = H(x)$ was defined in (1.2)]. Thus, we obtain the tensor-linear relations between σ_{kl} and η_{kl}:

$$\eta_{kl} = \frac{1}{n+1}\,\frac{\partial W}{\partial \sigma_{kl}} = \frac{B_0}{2}\,(\sigma_i^2 + c^0 I_\sigma^2)^{(n-1)/2}(3\sigma_{kl}^0 + 2c^0 I_\sigma \delta_{kl}).$$

Based on the data of torsion experiments $(\sigma_{12} = \tau)$, uniaxial tension $(\sigma_{11} = \sigma > 0)$, and uniaxial compression $(\sigma_{11} = \sigma < 0)$, by comparing Eqs. (5.1) with the corresponding expressions for W under the above-indicated types of the stressed state [12]: $W = B_{kp}(\sqrt{3}\tau)^{n+1}$, $W = B_+\sigma^{n+1}$, and $W = B_-|\sigma|^{n+1}$, we determine the constants B_0, c_+^0, and c_-^0. As a result, we obtain

$$B_0 = B_{kp}, \qquad c_\pm^0 = (B_\pm/B_0)^{2/(n+1)} - 1,$$

where B_{kp}, B_+, and B_- are the creep coefficients under torsion, tension, and compression, respectively.

It follows from Eqs. (5.1) that the dissipative function $W = W(\sigma_i, I_\sigma)$ for $n > 1$ [and, hence, the creep potential equal to $W/(n+1)$] is a twice continuously differentiable function everywhere, including the situation with $\sigma_i = I_\sigma = 0$. Therefore, the condition of stability in the large (2.2) is equivalent to the positive determinacy of a symmetric matrix with the coefficients a_{ij} from (2.3), i.e., to the condition of satisfaction of the inequalities $a_{11} > 0$, $a_{22} > 0$, and $a_{11}a_{33} - a_{13}^2 > 0$, which reduce to the expressions $B_0 > 0$, $c_+^0 > 0$, and $c_-^0 > 0$. Thus, we obtain the constraints on the creep coefficients: $B_+ > B_0$ and $B_- > B_0$.

This work was supported by the Russian Foundation for Basic Research (Grant Nos. 05-01-00673, 05-01-08025, and 06-08-96002).

REFERENCES

1. S. A. Ambartsumyan and A. A. Khachatryan, "Constitutive equations of the elasticity theory with different degrees of resistance to tension and compression," *Inzh. Zh., Mekh. Tverd. Tela*, No. 2, 44–53 (1966).
2. S. A. Ambartsumyan and A. A. Khachatryan, "On multimodulus elasticity theory," *Inzh. Zh., Mekh. Tverd. Tela*, No. 6, 64–67 (1966).
3. G. S. Shapiro, "Strains of solids with different degrees of resistance to tension and compression," *Inzh. Zh., Mekh. Tverd. Tela*, No. 2, 123–125 (1966).
4. N. M. Matchenko and L. A. Tolokonnikov, "Relations between stresses and strains in multimodulus isotropic media," *Inzh. Zh., Mekh. Tverd. Tela*, No. 6, 108–110 (1968).
5. E. V. Lomakin, "Constitutive relations of mechanics of multimodulus solids," Preprint No. 159, Inst. Probl. Mech., Acad. of Sci. of the USSR, Moscow (1980).
6. S. A. Ambartsumyan, *Multimodulus Elasticity Theory* [in Russian], Nauka, Moscow (1982).
7. I. Yu. Tsvelodub, "On multimodulus elasticity theory for isotropic materials," in: *Dynamics of Continuous Media* (collected scientific papers) [in Russian], No. 32, Inst. of Hydrodynamics, Sib. Div., Acad. of Sci. of the USSR, Novosibirsk (1977), pp. 123–131.
8. E. V. Lomakin and Yu. N. Rabotnov, "Relations of the elasticity theory for an isotropic multimodulus solid," *Izv. Akad. Nauk SSSR, Mekh. Tverd. Tela*, No. 6, 29–34 (178).
9. V. P. Myasnikov and A. I. Oleinikov, "Generalized constitutive relations of an isotropic elastic medium with different resistances," *Dokl. Akad. Nauk SSSR*, **332**, No. 1, 57–60 (1992).
10. A. A. Burenin and V. M. Yarushina, "Modeling of strains in materials with different resistances to tension and compression," in: *Problems of Mechanics of Deformable Solids and Rocks* (collected scientific papers devoted to the 75th anniversary of E. I. Shemyakin) [in Russian], Fizmatlit, Moscow (2006), pp. 100–106.
11. N. I. Muskhelishvili, *Some Basic Problems of the Mathematical Theory of Elasticity*, Noordhoff, Groningen (1963).
12. I. Yu. Tsvelodub, *Stability Postulate and Its Application to the Theory of Creep of Metals* [in Russian], Inst. of Hydrodynamics, Sib. Div., Russian Acad. of Sci., Novosibirsk (1991).
13. A. I. Oleinikov and E. V. Mogil'nikov, "On uniqueness of the solution of problems of stability of a material with nonlinear heterogeneous elasticity," *Dokl. Ross. Akad. Nauk*, **378**, No. 2, 194–196 (2001).